Fundamental of Shortest Path Algorithms

Kuan-Yu Chen (陳冠宇)

2019/06/04 @ TR-310-1, NTUST

Review

- They each use a specific rule to determine a safe edge in line 3 of GENERIC-MST

```
GENERIC-MST(G, w)
1  A = ∅
2  while A does not form a spanning tree
3      find an edge (u, v) that is safe for A
4      A = A ∪ {(u, v)}
5  return A
```

  - In Kruskal's algorithm
    - The set *A* is a forest whose vertices are all those of the given graph
    - The safe edge added to *A* is always a least-weight edge in the graph that **connects two distinct components**
  - In Prim's algorithm
    - The set *A* forms a single tree
    - The safe edge added to A is always a least-weight edge **connecting the tree to a vertex not in the tree**

Shortest-paths Problem

- In a ***shortest-paths problem***
  - Given a weighted directed graph $G = (V, E)$
    - The weight function $w: E \to \mathbb{R}$ mapping edges to real-valued weights
  - The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

  - We define the ***shortest-path weight*** $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p): u \overset{p}{\leadsto} v\}, & \mathit{if\ there\ is\ a\ path\ from\ u\ to\ v} \\ \infty & ,\mathit{otherwise} \end{cases}$$

  - We shall focus on the ***single-source shortest-paths problem***
    - Given a graph $G = (V, E)$, we want to find a shortest path from a given ***source*** vertex $s \in V$ to each vertex $v \in V$

Optimal Substructure.

- Shortest-paths algorithms typically rely on the property that a shortest path between two vertices contains other shortest paths within it
  - Lemma: ***Subpaths of shortest paths are shortest paths***
    - Given a weighted, directed graph $G = (V, E)$ with weight function $w: E \to \mathbb{R}$
    - Let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex v_0 to v_k
    - For any i and j such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath of p from vertex v_i to v_j
    - Then, p_{ij} is a shortest path from v_i to vertex v_j

Optimal Substructure..

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  - Let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex v_0 to v_k
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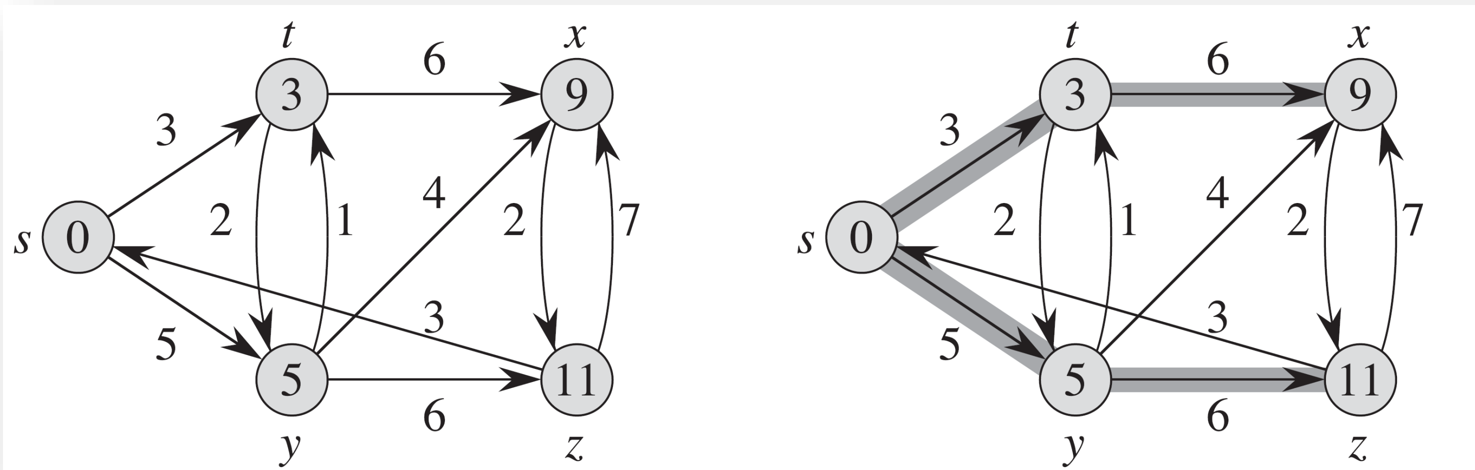
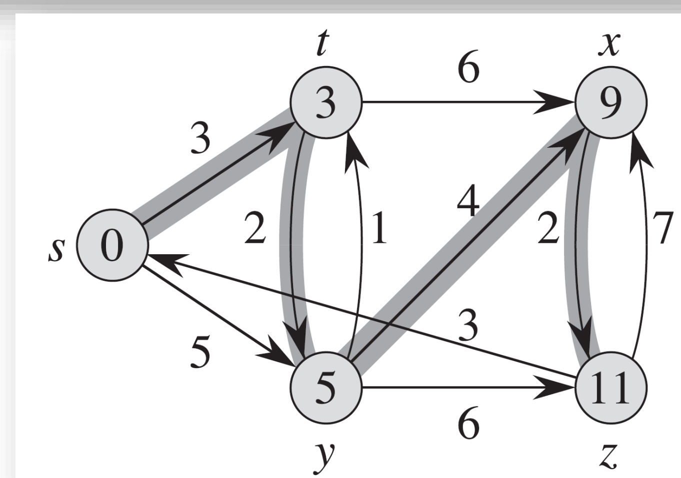
- Proof:
  - If we decompose path p into $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$, we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$
  - Assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$, then $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p'_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$ $\rightarrow\leftarrow$

Unique?

- Shortest paths are not necessarily unique

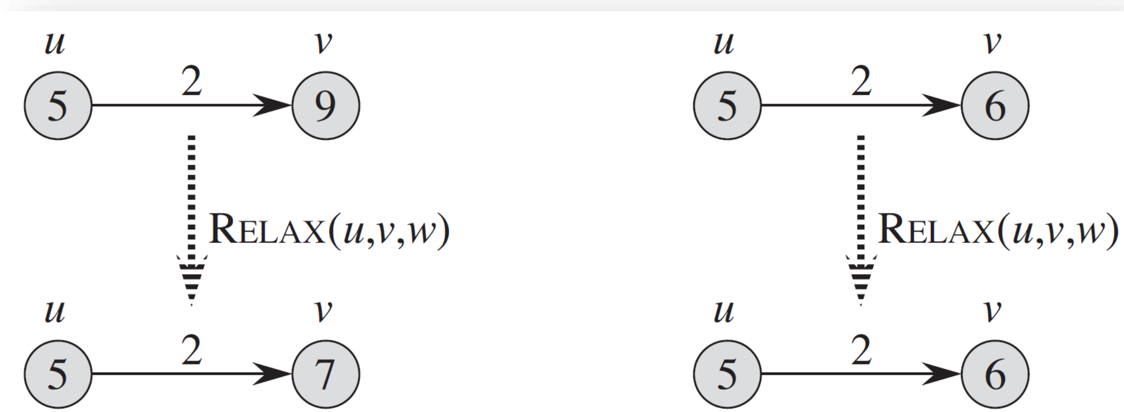




Relaxation

- For each vertex $v \in V$, we maintain an attribute $v.d$, which is an upper bound on the weight of a shortest path from source s to v
  - $v.d$ is a ***shortest-path estimate***

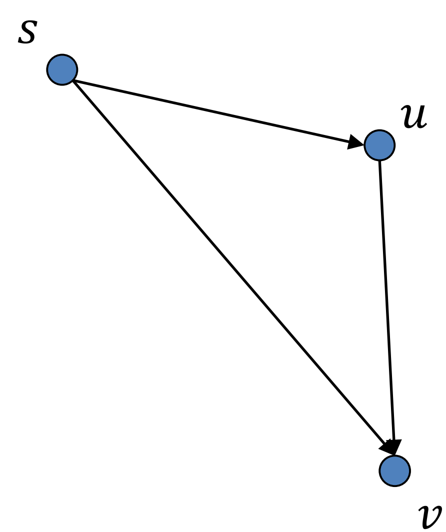
- The process of ***relaxing*** an edge (u, v) consists of testing whether we can improve the shortest path to v found so far by going through u and, if so, updating $v.d$
  - If $v.d > u.d + w(u, v)$, then $v.d = u.d + w(u, v)$




```
RELAX(u, v, w)
if v.d > u.d + w(u, v)
    v.d = u.d + w(u, v)
    v.π = u
```

Properties.

- **Triangle inequality**
  - Let $G = (V, E)$ be a weighted, directed graph with weight function $w: E \rightarrow \mathbb{R}$ and source vertex s. Then, for all edges $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$



Properties..

```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

- **Upper-bound property**
  - Let $G = (V, E)$ be a weighted, directed graph with weight function $w: E \rightarrow \mathbb{R}$. Let $s \in V$ be the source vertex, and let the graph be initialized by INITIALIZESINGLE-SOURCE(G, s). Then, $v.d \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once $v.d$ achieves its lower bound $\delta(s, v)$, it never changes.
  - Proof:
    - $v.d \geq \delta(s, v)$ is true after initialization for all $v \in V - \{s\}$
    - By considering the relaxation of edge (u, v), only $v.d$ may change
    - If $v.d$ changes, we have:

      $$v.d = u.d + w(u, v) \geq \delta(s, u) + w(u, v) \geq \delta(s, v)$$

      the invariant is maintained!
    - $v.d$ never changes once $v.d = \delta(s, v)$, since it achieves the lower-bound $\delta(s, v)$ and relaxation steps do not increase d value

Properties…

- **Convergence property**
  - Let $G = (V, E)$ be a weighted, directed graph with weight function $w: E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let $s \leadsto u \to v$ be a shortest path in G for some vertices $u, v \in V$. Suppose that G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps that includes the call RELAX(u, v, w) is executed on the edges of G. If $u.d = \delta(s, u)$ at any time prior to the call, then $v.d = \delta(s, v)$ at all times after the call.
  - Proof:
    - We are sure that $u.d = \delta(s, u)$ before and after we relax (u, v)
    - After relaxing edge (u, v), we have $v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v)$
    - By the upper-bound property, $v.d \geq \delta(s, v)$
    - Consequently, we conclude that $v.d = \delta(s, v)$

Properties….

- **Path-relaxation property**
  - Let $G = (V, E)$ be a weighted, directed graph with weight function $w: E \rightarrow \mathbb{R}$. Let $s \in V$ be the source vertex, and suppose that G is initialized by INITIALIZE-SINGLE-SOURCE(G, s). Consider any shortest path $p = \langle v_0, v_1, \dots v_k \rangle$ from $s = v_0$ to v_k, and then a sequence of relaxation steps occurs that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward.
    - This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of the edges of p

  - Proof:
    - For v_0, $v_0.d = 0 = \delta(s, s)$
    - By induction theorem, we first assume that $v_{i-1}.d = \delta(s, v_{i-1})$, then if we relax (v_{i-1}, v_i), we have $v_i.d = \delta(s, v_i)$ by convergence property

Properties

Triangle inequality

For any edge $(u, v) \in E$, we have $\delta(s, v) \le \delta(s, u) + w(u, v)$.

Upper-bound property

We always have $v.d \ge \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property

If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.

Convergence property

If $s \rightsquigarrow u \to v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property

If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Predecessor-subgraph property

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Questions?

kychen@mail.ntust.edu.tw